MATH3550 Real Analysis II
Final Mock

December 2026

There are five questions in this paper. You must complete all questions for full marks.
Duration: 2 HOURS

Section 1: Continuity

1. Recall Thomae's function (10)

$$t(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ 1 & \text{if } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ and } q > 0, \end{cases}$$

where $\gcd(p, q) = 1$. Prove that this function is discontinuous at every rational number.

Section 2: Derivatives

2. (a) State and prove the *Interior Extremum theorem*. (8)
   (b) Let a be a positive real number. Use the interior extremum theorem to prove that $f(x) = ax$ never attains a maximum on any open interval. (2)

Section 3: Sequences and series of functions

3. Recall the *Cauchy condition* for a family of functions $\big(f_n(x)\big)_{n\in\mathbb{N}}$, each defined on $A \subseteq \mathbb{R}$, says that for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon$$

for every $n, m > N$ and for every $x \in A$. Prove the *Cauchy criterion for uniform convergence* by completing the following two questions

   (a) Assume a family of functions $\big(f_n(x)\big)$ converges uniformly. Using the triangle inequality, or any other method, show that $\big(f_n(x)\big)$ satisfies the Cauchy condition. (5)

(b) Assume $\big(f_n(x)\big)$ satisfies the Cauchy condition. Prove that $\big(f_n(x)\big)$ converges uniformly. (5)

4. (a) State *Lagrange's Remainder Theorem.* (2)

(b) Calculate the *Maclaurin series* of $\cos(x)$. (4)

(c) Estimate the error of approximation to $\cos(x)$ of the function (4)

$$S_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

on the interval $[-2, 2]$.

Section 4: Integration

5. (a) Let $\big(f_n(x)\big)$ be a family of functions that are each Riemann-integrable on $[a, b] \subseteq \mathbb{R}$. Say $\big(f_n(x)\big)$ converges uniformly to $f(x)$ on this interval. Prove that $f(x)$ is Riemann-integrable on $[a, b]$. (8)

(b) Is the statement true when the functions $f_n(x)$ are not necessarily Riemann-integrable? Justify your answer with an example. (2)